ON THE PROPERTIES AND THE DETERMINATION OF THE WRENCH-CLOSURE WORKSPACE OF PLANAR PARALLEL CABLE-DRIVEN MECHANISMS

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ABSTRACT
This paper presents a detailed analysis of the constant-orientation wrench-closure workspace of planar three-degree-of-freedom parallel mechanisms driven by four cables. The constant-orientation wrench-closure workspace is defined as the subset of the plane wherein, for a given orientation of the moving platform, any planar wrench applied on the moving platform can be balanced by the cable-driven mechanism. Based on mathematical observations, this workspace is proved to be the union of two disconnected sets that may or may not exist. Moreover, if the constant-orientation wrench-closure workspace (WCW) exists, its boundary is shown to be composed of portions of conic sections. Then, an algorithm that determines the constant-orientation wrench-closure workspace by means of a graphical representation of its boundary is introduced. Several examples are also included.

1 INTRODUCTION
A planar parallel cable-driven mechanism consists essentially of a platform, moving in a plane, connected in parallel to a base by lightweight links such as cables or wires. Each cable is wound around an actuated reel fixed to the base and attached to the moving platform at its other end. The reels control the pose of the platform by controlling the length of their respective cable. The tension in the cables, also controlled by the reels, allows the platform to apply or withstand wrenches. The platform’s reachable space is ultimately limited by the total length of each cable that is by the constant sum of wound and unwound length of each cable. Large total lengths are easily stored on reels providing cable-driven mechanisms the possibility of working inside a very large space compared with those of conventional rigid-link parallel mechanisms. However, the cables can only pull on the platform. Consequently, depending on the task, a cable-driven mechanism often does not have the ability to work in all this large space. Indeed, there exists many poses inside this space for which the cables are unable to balance a given wrench because at least one of them has to push on the platform. Hence, the relationship between the pose and the feasible wrenches at the platform appears as a fundamental issue for cable-driven mechanisms even if the actuators are assumed to have no limits in strength and power. Defining the workspace as the set of poses for which any wrench of a given set of wrenches can be applied on the plat-
form by pulling the latter with the cables, gives a means to study some aspects of this relation in terms of workspace. Thereafter, a wrench is said to be feasible if none of the cables has to push on the platform to produce it.

In [1] and [2], the workspace of planar cable-driven mechanisms is studied as the set of poses for which a given planar wrench $w$ is feasible. In [1], it is called dynamic workspace because it depends on $w$, hence, during a motion, its shape changes together with the accelerations of the moving platform. A particular instance of this dynamic workspace is obtained by setting $w$ equal to the opposite of the wrench applied by gravity forces on the platform — or to zero in case of a zero-gravity environment. This particular case corresponds to the set of static equilibrium poses of the platform. It is the subject of design in [3] and in [4]. Conditions for a pose to be inside of this set of static equilibria are proposed in [5].

From a general design point of view, the workspace defined as the set of poses for which any wrench is feasible is of great interest. References [6–8] pointed out a necessary condition on the number of cables for this workspace to exist: the cable number must be greater than the number of degrees of freedom of the platform. Kurtz and Hayward [6] and Kawamura and Ito [7] noticed the strong similarity between cable-driven mechanisms and multifinger grasping systems with frictionless points of contact: as the cables can only pull on the platform, the fingers can only push on a body. In both cases, the force has a unidirectional nature. Hence, a concept called force-closure grasp, when applied to cable-driven mechanisms, leads to the previously stated necessary condition on the cable number. This parallel motivated the name wrench-closure workspace given in this paper to the set of poses for which any wrench is feasible. The wrench-closure workspace is also known as the set of fully constrained configurations [5] or simply as the workspace [9]. In [10], a mechanism is said to be manipulable when its pose belongs to the wrench-closure workspace.

In this paper, the problem of the determination of the constant-orientation workspace of planar parallel mechanisms driven by four cables is addressed. The four-cable case is of practical interest since it corresponds to mechanisms having the minimum cable number that allows the wrench-closure workspace to exist. A well-known necessary and sufficient condition [1,5–11] for a pose of the moving platform to be inside the wrench-closure workspace leads to a simple characterization of this workspace. Several properties of constant-orientation sections of the wrench-closure workspace are deduced from this characterization. Then, based on these properties, the notion of “geometric” algorithm [1,12,13] is used in order to obtain an algorithm that determines the wrench-closure workspace by means of the description of its boundary. Finally, examples of nonvanishing constant-orientation wrench-closure workspaces are presented. Examples of mechanism architectures leading to a vanishing wrench-closure workspace are pointed out as well.

2 WRENCH-CLOSURE WORKSPACE : GENERAL PROPERTIES

2.1 Architecture, Kinematic Modeling and Wrench Transmission

A planar 4-cable-driven mechanism is schematically shown in Fig. 1. Each cable is attached to the moving platform and winds at the base around an actuated reel. Let us assume that the point $A_i$ at which a cable winds around its reel is fixed relative to the base. Similarly, the attachment points, denoted $B_i$, are assumed to be fixed relative to the platform. The $i$th cable is tense between the points $A_i$ and $B_i$ and assumed to be a segment of the straight line $(A_iB_i)$. The contact points $A_i$ and $B_i$ are modeled as revolute joints whose axes are orthogonal to the mechanism plane. These assumptions lead to the kinematic modeling of cable-driven mechanisms shown in Fig. 2. The base frame with origin in $O$ and axes $x$ and $y$ is a fixed reference frame. Selecting a reference point $P$ on the platform and a direction $x'$ fixed relative to the platform, we obtain the moving frame $(P, x', y')$. Then, the position $p = [x, y]^T$ of the platform is given by the projection of vector $OP$ in the base frame. Its orientation $\phi$ is the angle between the fixed $x$-axis and the moving $x'$-axis. The taut length of cable $i$ is denoted $\rho_i$ and the projections of $OA_i$ and $PB_i$ in the base frame are respectively denoted $a_i$ and $b_i$.

Then, let us define the force-direction vector $d_i$ as the projection of unit vector $(1/\rho_i)\overrightarrow{B_iA_i}$ in the base frame:

$$d_i = \frac{1}{\rho_i} (a_i - b_i - p) = \frac{1}{\rho_i} (a_i^V - p) \quad (1)$$

with:

$$a_i^V = a_i - b_i \quad (2)$$

\[ \text{Fig. 2 Kinematic Modeling.} \]
where $\mathbf{a}_i^p$ is the position vector of the point $A_i^p$. The $A_i^p$ are important points because they are possible vertex of the constant-orientation Wrench-Closure Workspace (WCW). Vector $\mathbf{a}_i^p$ depends on the mechanism architecture and on its orientation $\phi$. When $\mathbf{p} = \mathbf{a}_i^p$, $p_i = 0$ — $B_i$ and $A_i$ coincide — and eq. (1) is no longer valid. In this case, $d_i$ is defined as the null vector. Note that the positions $\mathbf{p} = \mathbf{a}_i^p$, $i = 1$ to 4, must be avoided because the platform touches the base: the points $A_i^p$ are outside any workspace.

The wrench applied at $P$ by the tense cable $i$ is $t_iw_i$ with:

$$w_i = \begin{bmatrix} d_i \\ \det(b_i, d_i) \end{bmatrix}$$

being a unit planar wrench and $t_i$ being the tension in cable $i$. It is noted that $t_i$ is always nonnegative. The above notation det stands for the determinant. Note that, at $\mathbf{p} = \mathbf{a}_i^p$, the wrench $t_iw_i$ vanishes because, by definition, $d_i$ is the zero vector and the force applied at $B_i$ by the base on the platform is an unknown contact force no longer given by $t_iw_i$. The platform wrench at $P$, $w_p$, is the sum of the cable wrenches $t_iw_i$. This sum can be written in matrix form as:

$$\mathbf{Wt} = w_p$$

with:

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix} \quad \text{and} \quad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$$

where $\mathbf{t}$ is the vector of cable tensions and $\mathbf{W}$ the wrench matrix. The wrench matrix is a pose dependent rectangular 3 by 4 matrix. Therefore, the dimension of its nullspace is greater than or equal to one. Henceforth, for any vector $\mathbf{v}$, $\mathbf{v} > 0$, $\mathbf{v} \geq 0$ and $\mathbf{v} < 0$ mean that all the components of $\mathbf{v}$ are greater than zero, greater than or equal to zero and smaller than zero respectively.

### 2.2 Nullspace Characterization of the WCW

The WCW is the set of poses for which any platform wrench is feasible i.e. for which any platform wrench can be produced by tightening the four cables. Based on the modeling of the previous subsection 2.1, let us give a more precise definition of the WCW:

**Definition 1** The WCW is the set of poses where, for any platform wrench $w_p$ in $\mathbb{R}^3$, there exists at least one vector $\mathbf{t} \geq 0$ such that $\mathbf{Wt} = w_p$.

The following well-known theorem gives a necessary and sufficient condition for a pose to be inside the WCW:

**Theorem 1** A pose belongs to the WCW if and only if:

$$\text{rank}(\mathbf{W}) = 3$$

and

$$\exists \mathbf{z} \in \ker(\mathbf{W}) \text{ such that } \mathbf{z} > 0$$

where $\ker(\mathbf{W})$ is the nullspace of $\mathbf{W}$. A proof of this theorem is proposed in [5,9,10]. The full rank condition on the wrench matrix $\mathbf{W}$ is necessary, otherwise, if $\text{rank}(\mathbf{W}) < 3$, some platform wrench cannot be generated even if the cables could push on the platform. The necessary condition implies that, for a pose inside the WCW, the platform wrench can be set to zero while the four cables are tense. To understand the sufficient condition of Theorem 1, consider $\mathbf{Wt} = w_p$ as an underdetermined linear system of three equations, the four components of $\mathbf{t}$ being the unknowns. According to the full rank condition (6), a pseudoinverse $\mathbf{W}^+$ of $\mathbf{W}$ exists and a solution of $\mathbf{Wt} = w_p$ is:

$$\mathbf{t} = \mathbf{W}^+w_p + \lambda\mathbf{z}, \quad \lambda \in \mathbb{R}, \quad \mathbf{z} \in \ker(\mathbf{W}), \quad \mathbf{z} > 0$$

with $\mathbf{z} > 0$ according to condition (7). The sufficient condition of Theorem 1 holds because, for any platform wrench $w_p$, a sufficiently large $\lambda$ leads to a nonnegative solution $\mathbf{t}$. Note that, when a wrench matrix $\mathbf{W}$ satisfies the two conditions of Theorem 1, the set formed by its columns is called a “Vector Closure” in [7,11].

Now, a theorem giving an expression for the vectors of the wrench matrix nullspace is stated. This expression will bring insight into the nature of the WCW.

**Theorem 2** if $\mathbf{W}$ has full rank then $\ker(\mathbf{W}) = \text{span}(\mathbf{z}_0)$ with:

$$\mathbf{z}_0 = \begin{bmatrix} \det(\ldots) \\ -\det(\ldots) \\ -\det(\ldots) \\ -\det(\ldots) \end{bmatrix}$$

This theorem is proved in the appendix in a more general case. The $i$th component of $\mathbf{z}_0$, $z_{0i}$, is equal to $(-1)^{i+1}\det_{i}$ where $\det_{i}$ is the determinant of the square matrix obtained from the wrench matrix $\mathbf{W}$ by deleting its $i$th column. Vector $\mathbf{z}_0$ is a non-zero vector if and only if $\mathbf{W}$ has full rank. Thus combining Theorem 1 and Theorem 2 leads to:

**Theorem 3** A pose belongs to the WCW if and only if $\mathbf{z}_0 > 0$ or $\mathbf{z}_0 < 0$.

Theorem 3 was also known by Gallina and Rosati [10]. Their proof relies on Theorem 2 which was on the other hand not proved in their work. Here we provide a demonstration of Theorem 2 because, in our opinion, it is not a well-known result of linear algebra. Theorem 3 provides a means of testing if a pose belongs to the WCW. Hence, a discretization algorithm can be used as a method for determining the WCW. The result is a cloud of points that satisfy Theorem 3. This method is easily applied but requires intensive computation and large amounts of disk space for storing the point cloud. Moreover, the use of a discretization algorithm for computing the WCW does not provide any further insight into the properties of the WCW. A geometric description of the constant-orientation WCW boundaries is rather proposed here because such a description results in a better understanding of the workspace properties and in a more efficient algorithm for its determination. Furthermore, the geometric description allows
a better visualization of the constant-orientation WCW. Before describing the main properties of the WCW, the next subsection introduces a superset of it.

2.3 A Superset of the WCW: Force-Closure Workspace

First, let us define the Force-Closure Workspace (FCW).

Definition 2 The FCW is the set of poses where:

\[ \text{rank}(W_{12}) = 2 \]  \hspace{1cm} (10)

and

\[ \exists z_f \geq 0 \text{ such that } W_{12} z_f = 0 \]  \hspace{1cm} (11)

with \( z_f \in \mathbb{R}^4 \) and \( W_{12} \) is the matrix obtained from \( W \) by deleting its last row i.e. \( W_{12} = [d_1 \ d_2 \ d_3 \ d_4] \).

The WCW is a subset of the FCW because eq. (6) implies eq. (10) and eq. (7) implies eq. (11). For a constant orientation of the moving platform, the set of positions of \( P \) for which definition 2 holds is called the constant-orientation FCW. The remainder of this subsection deals with the constant-orientation FCW. Since two columns of \( w_{12} \), say \( d_j \) and \( d_k \), are linearly dependent if and only if \( P \) belongs to the straight line \( (A_i^w A_j^w) \) then there exists a vector \( p \) such that \( \text{rank}(W_{12}) < 2 \) only if the four points \( A_i^w \) are collinear. In the event of an alignment of the \( A_i^w \), if \( (L) \) denotes the straight line passing through these points, \( \text{rank}(W_{12}) < 2 \) if \( P \) lies on \( (L) \) and the condition (11) implies that \( P \) lies on \( (L) \). Thus the two conditions (10) and (11) cannot be satisfied at the same time and the constant-orientation FCW does not exist when the four points \( A_i^w \) are collinear. In this case, since the WCW is a subset of the FCW, the constant-orientation WCW does not exist either. The alignment of the four points \( A_i^w \) occurs only for particular cable-driven mechanism designs that can be easily avoided. Thereafter, the points \( A_i^w \) are assumed to be non-collinear and hence condition (10) is met for any pose of the moving platform.

Inside the constant-orientation FCW any planar force can be generated at the platform by appropriately tightening the four cables. This is obviously the reason of its name. From eq. (11), the nature of the constant-orientation FCW can be found. In fact, with eq. (1) and for \( p \neq a_i^w \) (\( i = 1 \) to 4):

\[ W_{12} z_f = 0 \]  \hspace{1cm} (12)

\[ \iff \sum_{i=1}^{4} \frac{z_{fi}}{\rho_i} (a_i^w - p) = 0 \]  \hspace{1cm} (13)

\[ \iff \sum_{i=1}^{4} \lambda_i a_i^w = p \]  \hspace{1cm} (14)

with:

\[ \lambda_i = \frac{z_{fi}}{\rho_i \Sigma}, \quad \Sigma = \sum_{j=1}^{4} \frac{z_{fj}}{\rho_j} \]  \hspace{1cm} (15)

\( z_{fi} \) being the \( i \)th component of \( z_f \). Since \( z_f > 0 \) and \( \rho_i > 0 \) for any \( i \) (\( p \neq a_i^w \)), the coefficients \( \lambda_i \) defined in eq. (15) have the following two properties:

\[ \forall i, \lambda_i > 0 \]  \hspace{1cm} (16)

\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \]  \hspace{1cm} (17)

Eq. (14) together with the properties (16) and (17) constitute a necessary and sufficient condition for a position \( p \) to belong to the constant-orientation FCW, hence they characterize it. The sufficient condition holds because, given a position \( p \) satisfying eq. (14) with the \( \lambda_i \) having the two properties (16) and (17), the components of a vector \( z_f \) such that \( W_{12} z_f = 0 \) are:

\[ z_{fi} = \rho_i \lambda_i c \]  \hspace{1cm} (18)

where \( c \) is an arbitrary positive constant. Now, \( p \) in eq. (14) with the \( \lambda_i \) having the two properties (17) and \( \lambda_i \geq 0 \) for any \( i \) instead of (16) is a convex combination of the \( a_i^w \) and the set of all these convex combinations is the convex hull of \( A = \{a_1^w, a_2^w, a_3^w, a_4^w\} \) denoted by \( \text{conv}A \) [14]. Since \( A \) is a finite subset of \( \mathbb{R}^2 \), \( \text{conv}A \) is a polygon whose vertices are among the elements of \( A \) — note that since the orientation of the platform is constant the \( a_i^w \) are constant vectors. From a topological point of view, \( \text{conv}A \) is a closed set and one can prove that the set of all the linear combinations of the \( a_i^w \) (eq. (14)) with the \( \lambda_i \) having the two properties (16) and (17) i.e. the constant-orientation FCW is an open set such that \( \text{conv}A \) is its closure\(^\dagger\). Let us sum up the main result in a theorem:

Theorem 4 The constant-orientation FCW is the interior of the largest two-dimensional convex polygon whose vertices are two or three elements of the set \( \{A_1^w, A_2^w, A_3^w, A_4^w\} \).

The interior of a polygon is the polygon itself but without its edges. Note that Theorem 4 does not make sense when the four possible vertices \( A_i^w \) are collinear. Fig. 3 gives an example of the constant-orientation FCW for a mechanism having a square base and a rectangular moving platform. As stated previously, the constant-orientation WCW is a subset of the constant-orientation FCW. This result was indirectly assumed by [2]. In the latter reference, the open convex hull of \( A \) is proved to be a superset of the constant-orientation dynamic workspace obtained for \( w_p = 0 \). But, for a given orientation \( \phi \), this dynamic workspace is defined as the set of positions of the platform for which \( w_p = 0 \) is feasible with no slack cables, i.e., with \( t > 0 \): the constant-orientation dynamic workspace obtained for \( w_p = 0 \) is identical to the constant-orientation WCW and the open convex hull of \( A \) is a superset of these two workspaces.

2.4 General Properties of the Constant-Orientation WCW

Property 1 For a given mechanism, the constant-orientation WCW may not exist.

\(^\dagger\)The closure of a set \( A \) is the smallest closed set containing \( A \).
That is, some mechanism designs lead to a vanishing WCW, i.e., whatever the orientation of the moving platform, the constant-orientation WCW is non-existent. Other designs have a constant-orientation WCW that vanishes for some orientations. Examples of non-existent WCW will be presented in an upcoming section.

**Property 2** The constant-orientation WCW is a bounded subset of the plane.

According to subsection 2.3, the constant-orientation WCW is a subset of the constant-orientation FCW and, by Theorem 4, the constant-orientation FCW is a bounded subset of the plane. Hence, the constant-orientation WCW is also a bounded set.

**Property 3** The boundary of the constant-orientation WCW is composed of sections of quadratic curves.

Each component $z_{0i}$ of the vector $z_0$ defined by eq. (9) turns out to have a quadratic form in terms of the coordinates $x$ and $y$ of the platform reference point $P$. This quadratic expression can be written in matrix form:

$$z_{0i} = (-1)^{i+1} \det_i = (-1)^{i+1} \left( \frac{1}{2} \mathbf{p}^T \mathbf{H}_i \mathbf{p} + f_i^T \mathbf{p} + q_{0i} \right)$$

with:

$$\mathbf{H}_i = \begin{bmatrix} 2q_{1i} & q_{3i} \\ q_{3i} & 2q_{2i} \end{bmatrix} \quad \text{and} \quad f_i = \begin{bmatrix} q_{4i} \\ q_{5i} \end{bmatrix}$$

(19)

Notice that the six coefficients $q_{ji}$ ($j = 1, \ldots, 6$) are functions of $\phi$ and of the geometric parameters of the cable-driven mechanism. The curve $C_i$, defined by $z_{0i} = 0$, is a conic section — referred to as a quadratic curve — that divides the plane into regions where $z_{0i}$ is positive or negative. The curve $C_i$ is also the locus of points where $\det_i$ is zero. Thus, it turns out to be the type II singularity locus of the $3RPR$ parallel mechanism whose base and platform, assuming $i = 1$ and referring to Fig. 2, consist of the triangles $A_2A_3A_4$ and $B_2B_3B_4$ respectively [15]. Depending on the values of the geometric parameters of the cable-driven mechanism and on the orientation of its moving platform, $C_i$ is either a hyperbola ($\det(\mathbf{H}_i) < 0$), a parabola ($\det(\mathbf{H}_i) = 0$) or an ellipse ($\det(\mathbf{H}_i) > 0$). For instance, if $C_i$ is a hyperbola then its two branches split the plane in three parts as shown in Fig. 4. 

Now, according to Theorem 3, the constant-orientation WCW, $\mathcal{W}$, can be written as an union of two intersections:

$$\mathcal{W} = \mathcal{W}^+ \bigcup \mathcal{W}^-$$

(21)

with:

$$\mathcal{W}^+ = \bigcap_{i=1}^4 \mathcal{W}_i^+ = \{ \mathbf{p} \in \mathbb{R}^2 \mid z_{0i} > 0 \}$$

(22)

and:

$$\mathcal{W}^- = \bigcap_{i=1}^4 \mathcal{W}_i^- = \{ \mathbf{p} \in \mathbb{R}^2 \mid z_{0i} < 0 \}$$

(23)

By definition:

$$\mathcal{W}_i^+ = \{ \mathbf{p} \in \mathbb{R}^2 \mid z_{0i} > 0 \}$$

(24)

and:

$$\mathcal{W}_i^- = \{ \mathbf{p} \in \mathbb{R}^2 \mid z_{0i} < 0 \}$$

(25)

Since the quadratic curve $C_i$ is the boundary of $\mathcal{W}_i^+$ and of $\mathcal{W}_i^-$, the boundary of $\mathcal{W}$ is composed of sections of quadratic curves, namely of sections of the curves $C_i$, $i = 1, 2, 3, 4$. Furthermore, the intersection point between two of the four quadratic curves $C_i$ is a possible vertex of the constant-orientation WCW, and, since $\mathcal{W}^+ \bigcap \mathcal{W}^- = \emptyset$, if neither $\mathcal{W}^+$ nor $\mathcal{W}^-$ are empty, $\mathcal{W}$ consists of two disjoint sets. An example of a constant-orientation WCW...
composed of two disjoint parts is shown in Fig. 5 where $\mathcal{W}^+$ is the smallest one. Notice that, $\mathcal{W}^+$ and $\mathcal{W}^-$ are disjoint but their boundaries have a common vertex, $A_5$, which, together with the point $A_6$, is a position of the platform leading to a rank deficient wrench matrix $\mathbf{W}$. In this example, since the points $B_3$ and $B_4$ of the platform are coincident (Fig. 6), the quadratic curves $C_1$ and $C_2$ have degenerated into two intersecting lines, two of these four lines being coincident.

**Property 4** The constant-orientation WCW is an open set.

The sets $\mathcal{W}^+_i$, $i = 1$ to $4$, are eight open sets. Then, since the union of two open sets is also an open set and the intersection of a finite number of open sets is again an open set, $\mathcal{W}$ is an open set, that is, the points on the boundary of $\mathcal{W}$ do not belong to $\mathcal{W}$.

**Property 5** The constant-orientation WCW is a singularity-free workspace.

This property holds by definition of the WCW. In the case of parallel manipulators with rigid links, the workspace is the set of poses attainable by the mobile platform without violating any mechanical constraint. In general, there exists a set of poses within this workspace for which one or more forces and moments cannot be generated at the mobile platform. Such a pose is a singularity of the second kind according to the classification adopted in [16] and corresponds to a rank deficient wrench matrix. These singularities must be avoided and hence they limit the workspace of the parallel manipulator. This problem does not exist for the WCW of parallel cable-driven mechanisms because by definition, within the WCW, every force and moment can be generated. In fact, in the case of a planar parallel mechanism actuated with four cables, the wrench matrix $\mathbf{W}$ becomes rank deficient if and only if the four quadratic curves $C_i$, $i = 1$ to $4$, have a common intersection point. The foregoing discussion on the nature of $\mathcal{W}$’s boundary shows that such an intersection point, if it exists, is outside of $\mathcal{W}$. As a matter of fact, even if this point is a vertex of $\mathcal{W}$ — for instance point $A_5$ in Fig. 5 — it is outside of $\mathcal{W}$ for $\mathcal{W}$ is an open set.

### 3. AN ALGORITHM FOR THE DETERMINATION OF THE CONSTANT-ORIENTATION WCW

In this section, the main steps of an algorithm that provides a graphical representation of the constant-orientation WCW are briefly presented. This representation is obtained by means of the determination of the constant-orientation WCW boundary as shown in Fig. 6. According to the results of the previous discussion, the boundary is composed of sections of quadratic curves. The algorithm consists essentially in finding all the sections that may be elements of the boundary and, then, in selecting the elements that are effectively parts of the boundary among all the possible ones. Similar algorithms have already been introduced in [12] and [1]. In the former reference, sections of the constant-orientation workspace of a class of 6-DOF parallel manipulators are obtained. These sections are specified through the description of their boundaries that are composed of circular arcs. In the latter reference, quadratic curves are intersected with each other to create a set of sections which are possible parts of the constant-orientation dynamic workspace boundary of a planar parallel cable-driven mechanism. Then, these sections are tested in order to determine the boundary of the dynamic workspace. Hence, the algorithm discussed here and the methods used in the aforementioned references share the same basic philosophy but the implementation details are quite different. The algorithm steps are now described.

**Step 1:** For each of the quadratic curves $C_i$ ($i = 1, \ldots, 4$), determine all the intersections with the three other quadratic curves. If the quadratic curve has degenerated into two intersecting straight lines, find also their intersection point.

![Fig. 5 Example of a constant-orientation WCW consisting of two disjoint parts.](image1)

![Fig. 6 Graphical representation of the constant-orientation WCW of Fig. 5.](image2)
Notice that, according to [17], the quadratic curve $C_i$ degenerates into two straight lines if and only if:

$$\frac{1}{2} t^T E^T H_i E t - q_{0i} \det(H_i) = 0 \quad \text{with} \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (26)$$

Generally speaking, two quadratic curves can have between 0 and 4 intersection points. These intersection points can be found by determining the roots of a polynomial of degree four. However, when the platform position is $A_i^P (P \equiv A_j^P)$, by definition, the force-direction vector $d_i$ is equal to the zero vector and, according to eq. (3), the unit planar wrench $w_i$ is also null. Hence, for $P \equiv A_j^P$, the three components of $z_0$ having $w_i$ in their expression in eq. (9) are equal to zero and it turns out that the three corresponding quadratic curves pass through $A_j^P$. Consequently, two quadratic curves $C_i$ and $C_j$ intersect each other in at least two points $A_i^P$ and $A_j^P$ — or in one point if $A_i^P$ and $A_j^P$ are coincident — that can be computed a priori by means of eq. (2). For instance, the quadratic curve $C_4$ passes through $A_1^P$, $A_2^P$ and $A_3^P$ whereas the curve $C_3$ passes through $A_1^P$, $A_2^P$ and $A_4^P$. Thus the two curves $C_3$ and $C_4$ intersect each other in $A_1^P$ and $A_2^P$.

**Step 2:** For each of the quadratic curves $C_i (i = 1, \ldots, 4)$, find all the sections defined by the intersection points of step 1.

A section is a quadratic curve segment whose extremities are two intersection points of step 1. The sections are found by ordering the intersection points. For example, in the case shown in Fig. 5, the sections of the curve $C_4$ are the segments $A_1^P A_2^P$, $A_2^P A_3^P$, $A_3^P A_4^P$, $A_4^P A_1^P$, $A_1^P A_4^P$, $A_2^P A_3^P$, $A_4^P A_2^P$ and $A_3^P A_4^P$. The next step aims at obtaining the sections which are effectively portions of the constant-orientation WCW boundary among all the potential candidates computed by step 2.

**Step 3:** Test each of the sections found in step 2 in order to obtain the sections that constitute the boundary of the constant-orientation WCW.

Since the constant-orientation WCW is a bounded subset of the plane, any infinite section is eliminated, for it cannot belong to the workspace boundary. An infinite section is a portion of a line, a parabola or a hyperbola that is limited by only one intersection point found in step 1. For instance, in Fig. 5, the hyperbola $C_3$ has four infinite sections, two per branch, and three finite sections $A_1^P A_2^P$, $A_2^P A_3^P$ and $A_3^P A_4^P$. A test is performed on one point of each of the finite sections. This point can be any point of the section located between its two extremities. In practice, the test point is chosen far enough from the two extremities. Notice that this test point belongs to only one of the quadratic curves, say $C_j$, otherwise it would be an intersection point found in step 1 that is one of the section extremities. The test consists in evaluating $z_{0i}$ at the test point with eq. (9). The component $z_{0i}$ of $z_0$ is equal to zero since the test point is a point of the curve $C_j$. For the section to be effectively a portion of the constant-orientation WCW, the other components must have the same sign:

$$z_{0i} > 0 \quad \text{or} \quad z_{0i} < 0, \quad \forall i \neq j, \quad i = 1, \ldots, 4 \quad (27)$$

Note that this test cannot be performed when the quadratic curve $C_j$ degenerates into two straight lines, one of these straight lines being shared with another quadratic curve $C_k$ that has also degenerated into two straight lines — for instance the curves $C_1$ and $C_4$ in Fig. 5. In fact, at the test point, $z_{0k} = 0$ — the same situation occurs when two different non-degenerated quadratic curves $C_j$ and $C_k$ are coincident but this case is unlikely. The test presented in eq. (27) can neither be performed when the quadratic curve $C_j$ is a parabola that degenerates into only one line. Indeed, in this case, the sign of $z_{0j}$ remains the same in all the plane i.e. $W^+$ or $W^-$ is the empty set. For these two particular cases, the sign of the elements of $z_0$ is tested at one or two points which are not on the curve $C_j$ but near enough so that, if the section being tested is effectively a portion of the boundary of the constant-orientation WCW, one of these two points belongs to the constant-orientation WCW. In order to obtain these new test points, the tangent line $(T)$ to $C_j$ at one of the points, say $Q$, of the portion of $C_j$ that is being tested is determined. Then, the straight line $(O)$ passing through $Q$ and orthogonal to the tangent line $(T)$ is obtained. $(T)$ divides $(O)$ into two half-lines, say $(O_1)$ and $(O_2)$. The intersection points between $(O_i)$ $(i = 1$ or 2) and the three quadratic curves $C_i (i \neq j, \ i = 1, \ldots, 4)$ are computed. If such intersection points exist, the new test point, say $T_i (i = 1$ or 2), is any of the points of $(O_i)$ located between $Q$ and the intersection point nearest to $Q$. If no such intersection point exists, no new test point is created. Now, if no new test point has been created, the portion of $C_j$ being tested is not a part of the boundary of the constant-orientation WCW for the constant-orientation WCW is a bounded set. If one or two new test points $T_i$ have been found, the portion of $C_j$ being tested is a part of the boundary of the constant-orientation WCW if the elements of $z_0$ have the same sign at the test point $T_i$ or at least at one of the two points $T_1$ and $T_2$ if two test points have been created.

**Step 4:** If the constant-orientation WCW exists, produce a graphical representation of it by drawing the sections that satisfy the test performed in step 3.

The next section presents examples of results obtained with this algorithm.

### 4 EXAMPLES

#### 4.1 Rod type platform and square base

The moving platform of this mechanism has only two cable attachment points i.e. the points $B_1$ and $B_4$ and the points $B_2$ and $B_3$ are respectively coincident. Its base is a square. This architecture has the interesting property of having its constant-orientation WCW identical to its constant-orientation FCW for $\phi = 0$ and $\phi = \pi$. Hence, for these two orientations of the moving platform, the constant-orientation WCW is as large as possible since it is a subset of the constant-orientation FCW. For instance, the constant-orientation WCW for $\phi = 0$ is shown in Fig. 7. For the other orientations of the moving platform, the constant-orientation WCW vanishes or is a convex polygon as shown in Fig. 8. Indeed, the constant-orientation WCW exists when the orientation $\phi$ is in the open intervals $(-\pi/4, \pi/4)$
Fig. 7 Rod type platform: constant-orientation WCW for $\phi = 0$ deg.

Fig. 8 Rod type platform: constant-orientation WCW for $\phi = 40$ deg.

Fig. 9 Rod type platform: WCW for $-40$ deg $\leq \phi \leq 40$ deg.

and $(3\pi/4, -3\pi/4)$. The former case is shown in Fig. 9. Note that, if three cables are attached to the moving platform at the same point, whatever the orientation of the moving platform, the constant-orientation WCW vanishes i.e. the WCW does not exist.

4.2 Rectangular platform and square base
The moving platform of this mechanism is a rectangle and its base is a square. The constant-orientation WCW for $\phi = 10$ deg is shown in Fig. 10. In this case, it is composed of two portions of hyperbolas. For the mechanism shown in Fig. 10, the constant-orientation WCW exists when the orientation $\phi$ is in the open intervals $(-\pi/12, \pi/12)$ and $(11\pi/12, -11\pi/12)$. The former case is shown in Fig. 11. As for the mechanism with the rod type platform, this mechanism has the interesting property of having its constant-orientation WCW identical to its constant-orientation FCW for $\phi = 0$ and $\phi = \pi$. Note that, if the moving platform and the base are squares, whatever their dimensions, the WCW does not exist.

5 CONCLUSION
Several fundamental properties of the constant-orientation WCW of planar parallel mechanisms driven by four cables have been presented. It has been pointed out that this workspace may be non-existent. If it exists, the constant-orientation WCW was proved to be a bounded open subset of the plane which is pos-
sibly composed of two disjoint sets. Moreover, the nature of its boundary has been disclosed. Indeed the boundary is a finite set of portions of conic sections. Based on this geometric property, an algorithm that determines the constant-orientation WCW of any planar parallel mechanism driven by four cables has been introduced. Since the WCW of such mechanisms is a rather small workspace, this algorithm is an important and useful tool for the design. Future research will address the problem of the determination of the the constant-orientation WCW of planar parallel mechanisms driven by more than four cables.

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REFERENCES


7 APPENDIX

Let \( W = [w_1 \ldots w_n w_{n+1}] \) be a \( n \) by \( (n+1) \) matrix having full rank \( n \). Hence \( \ker(W) = \text{span}(z_0) \) with:

\[
\begin{bmatrix}
\det([w_{n+1} \ldots w_n]) \\
\det([w_1 \ldots w_{n+1}]) \\
\vdots \\
-\det([w_1 \ldots w_n])
\end{bmatrix}_{n+1}
\]

\( z_0 \)

Proof:

According to a classic theorem of linear algebra:

\[
\dim(\ker(W)) = (n+1) - \text{rank}(W)
\]

Thus \( \dim(\ker(W)) = 1 \) because \( \text{rank}(W) = n \). Consequently, for any non-zero vector \( z \) of the nullspace of \( W \):

\[
\ker(W) = \text{span}(z)
\]

Now, since \( W \) has full rank, the vector \( z_0 \) defined by eq. (28) is not null and we may assume that \( \det([w_1 \ldots w_n]) \neq 0 \) i.e. that \( (w_1, w_2, \ldots, w_n) \) is a basis of \( \mathbb{R}^n \). Hence:

\[
\exists (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n \mid w_{n+1} = \sum_{j=1}^{n} \beta_j w_j
\]

but:

\[
Wz_0 = \sum_{i=1}^{n} w_i \det([w_1 \ldots w_{i-1} w_{n+1} w_{i+1} \ldots w_n]) \\
-\det([w_1 \ldots w_n])
\]

The use of eq. (31) and of classic properties of determinants leads to:

\[
\det([w_1 \ldots w_{i-1} w_{n+1} w_{i+1} \ldots w_n]) = \beta_i \det([w_1 \ldots w_{i-1} \sum_{j=1}^{n} \beta_j w_j w_{i+1} \ldots w_n])
\]

hence with eq. (32):

\[
Wz_0 = \det([w_1 \ldots w_n]) \left( \sum_{i=1}^{n} \beta_i w_i - w_{n+1} \right)
\]

Finally, from eq. (31):

\[
Wz_0 = 0
\]

i.e. \( z_0 \) belongs to the nullspace of \( W \) and, since \( z_0 \) is not null, \( \ker(W) = \text{span}(z_0) \).